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ИЗУЧЕНИЯ НЕЛИНЕЙНАЯ ДИФФУЗИЯ В СВЕРХПРОВОДНИКАХ

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Аннотация. *В данной работе динамика проникновения магнитного потока в сверхпроводниках в вязком режиме течения исследуется путем аналитического решения уравнения нелинейной диффузии для индукции магнитного поля. Предполагается, что приложенное внешнее поле параллельно поверхности образца. Точное решение нелинейного уравнения диффузии для магнитной индукции было получено с использованием хорошо известного метода масштабирования.*

Ключевые слова: *сверхпроводники, магнитное поле, малые возмущения, критическое состояние .*

O'ta-o'tkazgichlarda nochiziqli diffuziya hodisalarini o'rganish

Annotatsiya. *Ushbu maqolada yopishqoq oqim rejimida magnit oqimning dinamikasi magnit maydon induksiyasi uchun chiziqli bo'lmagan diffuziya tenglamasini analitik echish yo'li bilan o'rganilgan. Tashqi maydon namuna yuzasiga parallel deb hisoblangan. Magnit induksiyasi uchun nochiziqli diffuziya tenglamasi o'xshashlik tamoyiliga asoslanib topilgan.*

Kalit so'zlar *: o'ta-o'tkazuvchilar, magnit maydon, oqim, diffuziya, kritir holat.*

Nonlinear diffusion in superconductors

Abstract. *In this paper, the dynamics of magnetic flux penetration in a viscous flow regime is studied by analytically solving the nonlinear diffusion equation for the magnetic field induction. It was assumed that the applied external field is parallel to the sample surface. An exact solution of the nonlinear diffusion equation for the magnetic induction* $\vec{B}(\vec{r},t)$ *has been obtained using a well-known scaling method.*

Key words: *superconductors, magnetic field, small perturbations, critical state.*

INTRODUCTION

The study of dynamics of the evolution of magnetic flux into a superconductor is an important problem of technical superconductivity. Mathematically, this problem can be formulated on the basis of a system of nonlinear evolutionary equations for the electromagnetic field, taking into account the relationship between the field and current in a superconductor [1]. In this paper, the dynamics of magnetic flux penetration in a viscous flow regime is studied by analytically solving the nonlinear diffusion equation for the magnetic field induction. It is assumed that the applied external field is parallel to the sample surface. An exact solution of the nonlinear diffusion equation for the magnetic induction $\vec{B}(\vec{r}, t)$ has been obtained using a well-known scaling method [2]. The problem is investigated within the framework of the macroscopic approach [3], in which all length scales exceed the distance between the flow lines; thus, a superconductor is considered as a homogeneous medium. It is shown that the velocity of propagation and the depth of penetration of the flux depend on the amplitude of the magnetic field on the surface of the superconductor, the critical current density, and the differential resistivity of the sample.

MODEL OF VISCOUS FLUX FLOW REGIME

Let us consider the regime of viscous flux flow, which can be represented as follows: if a transport current is passed through the superconductor, then the interaction of vortices with the current leads to the appearance of the Lorentz force acting on the vortex filament

$$\vec{F}_L = \frac{1}{c}\left[\vec{j}, \vec{\Phi}_0\right], \qquad (1)$$

where c is the speed of light, Φ_0 is the quantum of the magnetic flux. Under the influence of the Lorentz force, the magnetic flux starts to move, which causes energy dissipation, as a result of which the superconductor passes into a resistive state, or the Shubnikov phase. If there is a strong connection between the magnetic flux (Abrikosov vortex lattice) and the metal lattice, then the vortex lattice moves when $\vec{F}_L \geq \vec{F}_p$, where F_p is the pinning force. Thus, a viscous vortex flow regime is established in a superconductor. It follows from the definition of the pinning force that

$$j = j_c + \eta \frac{\nu c}{\Phi_0}, \qquad (2)$$

where η is the viscosity, v is the macroscopic velocity of the vortices. It is important that there is a functional connection between the velocity of vortices v and the change in the electric field E, which occurs when the magnetic flux moves. Indeed, consider the continuity equation for the flow of vortex filaments

$$\frac{dn}{dt} = \operatorname{div}(n \cdot v), \qquad (3)$$

(where n is the density of vortices under equilibrium conditions $B = n\Phi_0$) and Maxwell's equation

$$\frac{1}{c}\frac{dB}{dt} + \frac{dE}{dx} = 0. \qquad (4)$$

From (2) and (3) equations follows a functional connection

$$v = \frac{1}{c}\frac{E}{B}, \tag{5}$$

which plays an important role in the derivation of the equation for the evolution of the magnetic field. The equation of state for a superconductor has the form

$$j = j_c + \sigma_f E, \tag{6}$$

where $\sigma_f = \frac{1}{r_f}$ is the differential conductivity of the sample. From (6) it follows that the current-voltage characteristic in the regime of viscous flow of vortices can be written in the form

$$E = \rho_f(B)(j - j_c). \tag{7}$$

where $j_c = j_c(B,T)$. Below we obtain an equation that describes the distribution of the magnetic field induction, when there is a nonlinear dependence of the critical current on the magnetic field. Suppose the temperature of the superconductor is the same as the temperature of the cooling, i.e. we neglect the non-isothermality of the process, proceeding from the fact that the value of the diffusion coefficient provides a rapid equalization of the temperature gradient. Such efficient cooling takes place for composite superconductors. Under isothermal conditions, temperature can be considered as a parameter and, therefore, the relationship between magnetic induction, electric field and transport current is determined by the system of Maxwell's equations.

FORMULATION OF THE PROBLEM

Bean [4] has proposed the critical state model which is successfully used to describe magnetic properties of type II superconductors. According to this model, the distribution of the magnetic flux density $\vec{B}(r,t)$ and the transport current density $\vec{j}(r,t)$ inside a superconductor is given by the equation

$$\mathrm{rot}\vec{B} = \mu_0 \vec{j}. \tag{8}$$

When the penetrated magnetic flux changes with time, an electric field $\vec{E}(r,t)$ is generated inside the sample according to Faraday’s law

$$\mathrm{rot}\vec{E} = \frac{d\vec{B}}{dt}. \tag{9}$$

In the flux flow regime the electric field $\vec{E}(r,t)$ induced by moving of vortices is related with the local current density $\vec{j}(r,t)$ by the nonlinear Ohm’s law

$$\vec{E}=\rho\vec{j}. \tag{10}$$

In combining the relation (10) with Maxwell's equation (8) and (9), we obtain a nonlinear diffusion equation for the magnetic flux induction $\vec{B}(r,t)$ in the following form

$$\frac{d\vec{B}}{dt}=\frac{1}{\mu_0}\nabla\left[\rho(B)\nabla\vec{B}\right]. \tag{11}$$

Formally, this differential equation is simply a nonlinear diffusion equation with a diffusion coefficient depending on magnetic induction B. The parabolic type diffusion equation (11) allows to obtain a time and space distribution of the magnetic induction profile in a superconductor sample. It is evident that the space-time structure of solution of the diffusion equation (11) is determined by the characteristics of dependence of the differential resistivity ρ on the magnetic field induction B. Usually, in real experimental situation [5], the differential resistivity $\rho(B)$ grows with an increase of magnetic field induction

$$\rho=\frac{\Phi_0}{\eta c^2}\vec{B}=\rho_n\frac{\vec{B}}{B_{C2}} \tag{12}$$

where ρ_n is the differential resistivity in the normal state; η is the viscosity coefficient, $\Phi_0=\frac{hc}{2e}$ is the magnetic flux quantum, B_{C2} is the upper critical field of superconductor [5]. In the case, when the differential resistivity ρ is a linear function of the magnetic field induction B an exact solution of the diffusion equation (11) can be easily obtained by using the well-known scaling methods [2]. For the complex dependence of ρ(B) it can be use by empirical power-law dependence $\rho(B)\approx B^n$, where n is the positive constant parameter.

BASIC EQUATIONS

Now, we formulate the general equation governing the dynamics of the magnetic field induction in a superconductor sample. We study the evolution of the magnetic penetration process in a simple geometry - superconducting semi-infinitive sample x>0. We assume that the external magnetic field induction $\vec{B}_e$ is parallel to the z-axis. When the magnetic field with the flux density $\vec{B}_e(t)$ is applied in the direction of the z-axis, the transport current $\vec{j}(r,t)$ and the electric field $\vec{E}(r,t)$ are induced inside the slab along the y-axis. For this geometry, the spatial and temporal evolution of magnetic field induction $\vec{B}(r,t)$ is described by the following nonlinear diffusion equation in the generalized dimensionless form

$$\frac{db}{dt}=\frac{d}{dx}\left(b^n\left[\frac{db}{dx}\right]^q\right),\qquad (13)$$

where we have introduced the dimensionless parameters and variables $b=\frac{B}{B_e}$, $x_p=\frac{\mu_0 j_0}{B_e}x$, $t=\frac{t}{\tau_0}$, $j=\frac{j}{j_0}$, $B_e=\mu_0 j_0 v_0 \tau$.; $x_0=\frac{B_e}{\rho_0 j_c}$ is the magnetic field penetration depth; $\tau=\rho_n\frac{j_c^2\mu_0}{B_e^2}$ is the relaxation diffusion time; q is the positive constant parameter. The diffusion equation (13) can be integrated analytically subject to appropriate initial and boundary conditions in the center of the sample and on the sample's edges. We consider the case, when the magnetic field applied to sample increases with time according to a power law with the exponent of $\alpha > 0$

$$b(0,t)=b_0 t^{\alpha}.\qquad (14)$$

Boundary condition (14) is equivalent to a linear increase in the magnetic field with time, which corresponds to a real experimental situation [5]. As can be easily seen that the case α=0 describes a constant applied magnetic field at the surface of the sample, while the case α=1 corresponds to linearly increasing applied field, respectively. The other boundary condition follows from the continuity of the flux
at the free boundary $x=x_p$

$$b(x_p,t)=0,\qquad (15)$$

where x_p is the dimensionless position of the front of the magnetic field. The flux conservation condition for the magnetic field induction can be formulated in the following integral form

$$\int b(x,\,0)dx=1.\qquad (16)$$

It should be noted that the nonlinear diffusion equation (13), completed by the boundary conditions for magnetic induction, totally determines the problem of the space-time distribution of the magnetic flux penetration into superconductor sample in the flux flow regime with a power-law dependence of differential resistivity on the magnetic field induction. Solution of this equation gives a complete description of the time and space evolution of the magnetic flux in a sample.

SCALING SOLUTION

In the following analysis we derive an evolution equation for the magnetic induction profile and formulate a similarity solution for the $b(x,t)$. As can be shown that the nonlinear diffusion equation (13) can be solved exactly using well known scaling methods [2, 6]. At long times we present a solution of the nonlinear diffusion equation for the magnetic induction (13) in the following scaling form

$$\int b(x,\ t)dx = t^{\alpha} f\left(\frac{x}{t^{\beta}}\right). \qquad (17)$$

The similarity exponents α and β are of primary physical importance, since the parameter α represents the rate of decay of the magnetic induction b(x, t), while the parameter β is the rate of spread of the space distribution as time goes on. Inserting this scaling form into differential equation (13) and comparing powers of t in all terms, we get the following relationship for the exponents α and β

$$\alpha + 1 = \alpha(n+q) + \beta(1+q). \qquad (18)$$

Using the condition of the flux conservation (16) we obtain

$$\alpha = \beta = -\frac{1}{n+2q}, \qquad (19)$$

which suggests the existence of self-similar solutions in the form

$$b(z) = t^{-\frac{1}{n+2q}} f(z), \qquad z = \frac{x}{t^{\frac{1}{n+2q}}}. \qquad (20)$$

Substituting this scaling solution (20) into the governing equation (13) yields an ordinary differential equation for the scaling function f(z) in the form

$$(n+2q)\frac{d}{dz}\left[f^{n}\left(\frac{df}{dz}\right)^{q}\right] + z\frac{df}{dz} + f = 0. \qquad (21)$$

The boundary conditions for the function f(z) now become

$$f(0) = 1, \quad f(z_0) = 0. \qquad (22)$$

The above equation (21), depending on the initial and the boundary conditions describes a scaling—like behavior magnetic flux front with a time—dependent velocity in the sample. After a further integration and applying the boundary conditions (22) we get the following solution of the problem

$$f(z) = f(z_0)\left[1 - \left(\frac{z}{z_0}\right)^{1+q}\right]^{1/(n+q-1)}, \qquad (23)$$

$$f(z_0)=\left[n\frac{n+q-1}{1+q}\left(\frac{1}{n+2q}\right)^{1/q}\right]^{q/(n+q-1)} z_0^{(q+1)/(n+q-1)}.$$

The position of the front z_0 can now be found by substituting the solution (23) into the integral condition (16) and it is given by

$$z_0^{(n+2q)/(n+q-1)}=\frac{\left[\frac{n+q-1}{1+q}\left(\frac{1}{n+2q}\right)^{1/q}\right]^{q/(n+q-1)}}{\frac{1}{q+1}\frac{\Gamma\left(\frac{n+q}{n+q-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+q}{n+q-1}\right)\Gamma\left(\frac{1}{q+1}\right)}}.$$

It is convenient to write the self-similar solution (23) in terms of a primitive variables, as

$$b(x,t)=b_0\left[1-\left(\frac{x}{x_p}\right)^{(1+q)}\right]^{1/(n+q-1)}, \qquad (24)$$

where

$$b_0(0,t)=t^{-1/(n+2q)}\left[\frac{n+q-1}{1+q}\left(\frac{z_0^{(q+1)/q}}{n+2q}\right)^{1/q}\right]^{q/(n+q-1)}.$$

The profile of the the normalized flux density b(x, t) is shown schematically in figure 1.

Fig.1. The profile of the distribution of the normalized flux density b(x, t) at different times t=0.1, 0.2 for n=1.

This solution describes the propagation of the magnetic field into the sample, the magnetic induction being localized in the domain between the surface x=0 and the flux front x_p. This solution is positive in the plane $x_p^2 > x^2$ and is zero outside of it. Note, that only the x > 0 and t > 0 quarter of the plane is presented, because of it has physical relevance. The penetrating flux front position $x_p = x(t)$ as a function

of time can be described by the relation

$$x_p = \frac{z_0}{t^{\frac{1}{n+2q}}}. \qquad (25)$$

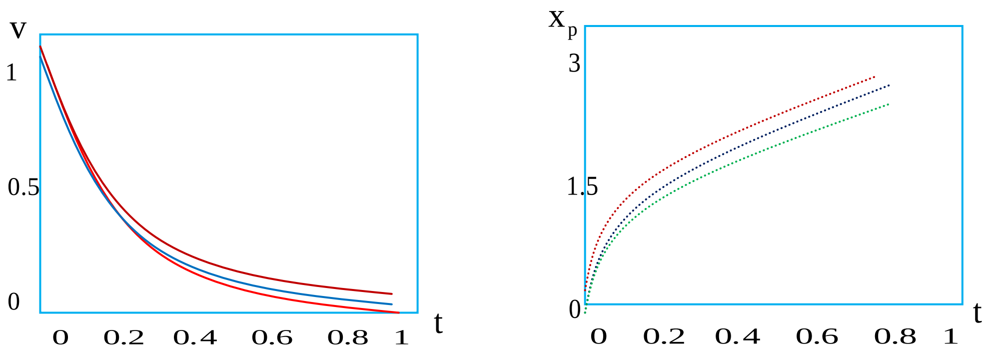

Fig.2. The profile of the magnetic flux front velocity at different values of n=3, 7, 11.

The velocity of the magnetic flux front decreases rapidly as the magnetic flux propagates (Fig2).

$$v_P(t) \equiv \frac{dx_p}{dt} \propto t^{-\frac{(2q+n-1)}{(n+2q)}}. \qquad (26)$$

The spatial and temporal profiles of magnetic flux penetration in the sample depends on the set of three independent parameters, n, q and α. It is of interest to consider the nonlinear diffusion equation for the magnetic induction for different values of the exponents n, q and α. For a given parameter set n, q and α the form of the scaling function f(z) can be obtained by solving the nonlinear diffusion equation (13) analytically by a self-similar technique. We solve the nonlinear diffusion equation analytically to provide expressions for the time-space evolution of the magnetic induction for different values of exponents n, q and α. Next, we analyze the effect of different values of exponents on the shape of the magnetic flux front in the sample. Varying the parameters of the equation, we may observe a various shapes of the magnetic flux front in the sample. As can be shown that different values exponents n and q generate different space–time magnetic flux fronts. Below we consider a few more practically relevant examples for which the magnetic flux front has a different shape depending on the different values of exponents n and q.

Let us first consider the most interesting case q=1. In this particular case the spatial and temporal evolution of the magnetic flux induction is totally determined by the parameters n and α. In the following analysis we derive an evolution equation for the magnetic induction profile and apply the scalings of the previous section to formulate a similarity solution for the b(x, t). For this particular case nonlinear diffusion equation (13) can be solved exactly using the scaling method. Thus, based on the scalings described in the previous section, we get the following relation for the exponents

$$\alpha=\beta=-\frac{1}{n+2}.$$

The last relation suggests the existence of solution to equation (13) of the form

$$b(x,t)=t^{-\frac{1}{n+2}}f(z); \qquad z=xt^{-\frac{1}{n+2}}. \tag{27}$$

Substituting the similarity solution (27) into the governing equation (13) yields an ordinary differential equation for the scaling function f(z)

$$(2+n)\frac{d}{dz}\left[f^n\frac{df}{dz}\right]+z\frac{df}{dz}+f=0. \tag{28}$$

Integrating the equation (28) by parts and applying the boundary conditions (16) give

$$f(z)=\left[\frac{n}{2(n+2)}z_0^2\right]^{1/n}\left[1-\left(\frac{z}{z_0}\right)^2\right]^{1/n}, \tag{29}$$

which is the explicit form of the similarity solution, which we have been seeking. The position of the front z_0 can now be found by substituting the last solution into the integral condition (16), so we have

$$\left[\frac{n}{2(n+2)}z_0^2\right]^{1/n}\int_0^{z_0}\left[1-\left(\frac{z}{z_0}\right)^2\right]^{1/n}dz=1. \tag{30}$$

By using the following transformation

$$z=z_0\sin\omega$$

and after integrating we obtain

$$z_0^{(n+2)/n}\left[\frac{n}{2(n+2)}\right]^{1/n}=\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}+\frac{1}{n}\right)}{\Gamma\left(1+\frac{1}{n}\right)}.$$

It is convenient to write the self-similar solution (29) in terms of a primitive variables, as

$$b(x,t)=b_0\left[1-\left(\frac{x}{x_0}\right)^2\right]^{1/n}, \qquad (31)$$

where

$$b_0=\left[\frac{n}{2(n+2)}z_0^2\right]^{1/n}t^{-\frac{1}{n+2}}. \qquad (32)$$

Equation (31) constitutes an exact solution of the nonlinear flux diffusion equation for the situation, when q=1. As can be seen the solution (31) describes the propagation of the flux profile inside the sample for the case q=1 is well. The profile of the the normalized flux density b(x, t) is shown schematically in figure 3.

Fig. 3. The distribution of the normalized flux density b(x, t) at different times t=0.1, 0.2 for n=1, q=1.

The penetrating flux front position $x=x_p(t)$ as a function of time can be described by the relation

$$x_p=z_0t^{\frac{1}{(n+2)}}.$$

The velocity of penetration of a magnetic flux into a superconductor can be naturally determined from the last relation

$$v \propto t^{-\frac{(n+1)}{(n+2)}}.$$

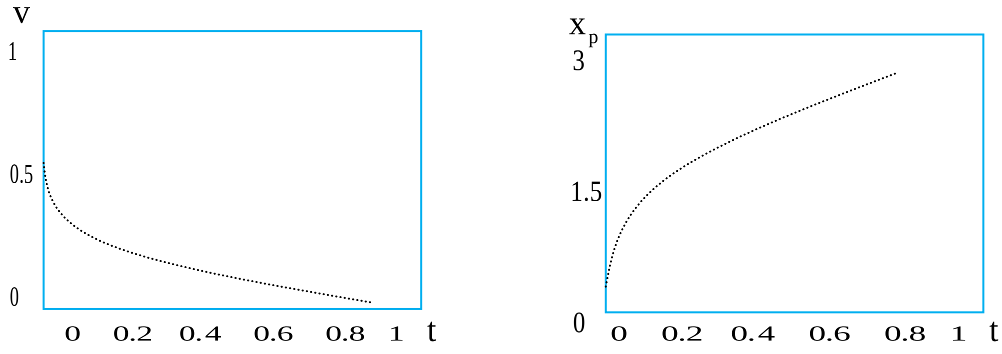

Fig 4. a) The profile of the velocity of the magnetic flux front at n = 1.
b) The time evolution of the magnetic flux for n = 1 and q=1.

Interestingly, the normalized current density $j(x,t)$ in the region, $0 < x < x_p$ can be presented using the equations

$$j(x,t) = -\frac{1}{c}\frac{db}{dx}. \tag{33}$$

After a simple analytical calculation, we can easily obtain the space and time profiles of the normalized current density $j(x,t)$.The space ant time evolution of the normalised current density is shown in Fig. 5 for the particular cases n = 1, 3.

Fig.5. The space ant time evolution of the normalised current density $j(x,t)$ for n = 1, 3.

Note that a similar problem was also studied in [7] in connection with the magnetic relaxation of a superconducting slab in the flux creep regime with power-law dependence of the electric field E on current density j. The authors of [7] showed that in the case of logarithmic barriers, the relaxation process leads to the self-organization of the system into the critical state. Assuming that the uniform magnetic induction B_0 is induced by a constant magnetic field, they found an expression for the magnetic moment for the limiting cfse $n = 1$. For example, it was shown that the pinning potential, which depends on the logarithmic dependence on the current density, leads to a similar nonlinear diffusion equation for the space-time evolution of the flux density with a power-law current-voltage characteristic [7].

CONCLUSION

In this paper, the dynamics of magnetic flux penetration in a viscous flow regime is studied by analytically solving the nonlinear diffusion equation for the magnetic field induction, assuming that the applied external field is parallel to the sample surface. An exact solution of the nonlinear diffusion equation for the magnetic induction $\vec{B}(\vec{r},t)$ has been obtained using a well-known scaling method. It was shown that the spatial and temporal profiles of magnetic flux penetration in the sample depends on the set of three independent parameters, n, q and α. For a given parameter set n, q and α the form of the scaling function f(z) is obtained by solving the nonlinear diffusion equation analytically by a self-similar technique. Next, we have analyzed the effect of different values of exponents on the shape of the magnetic flux front in the sample. Varying the parameters of the equation, we may observe a various shapes of the magnetic flux front in the sample. It was shown that different values exponents n and q generate different space–time magnetic flux fronts.

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