NONLINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATION WITH DEGENERATE KERNEL AND NONLINEAR MAXIMA

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Annotation: The initial value problem of solvability and construction of solutions of a nonlinear Fredholm integro-differential equation of first order with degenerate kernel and nonlinear maxima are considered. Using the method of degenerate kernel in combination it with the method of regularization, we obtain an implicit functional-differential equation of first order with nonlinear maxima. We use initial boundary conditions to ensure the uniqueness of the solution. In order to use the method of a successive approximations and prove the one value solvability, we transform the obtained implicit functional-differential equation to the nonlinear Volterra type integro-differential equation with nonlinear maxima. The one value solvability of the problem is proved.

Keywords: Integro-differential equation, first order, nonlinear functional-differential equation, degenerate kernel, nonlinear maxima, regularization, one value solvability.

Formulation of the problem

Integro-differential equations as integral and differential equations are mathematical models of the many physical processes and the operation in technical systems. In application of integro-differential equations the analytical and iterative methods play an important role [1-8].

In this paper, we study the initial value problem of one value solvability and construction of solutions of a nonlinear Fredholm integro-differential equation of first order with degenerate kernel and nonlinear maxima. When a kernel of integral is degenerate, it is easy to replace the given equation by implicit differential equation, which is convenient to transform into Volterra integro-differential equation for solving by the method of successive approximations. The integral and integro-differential equations with degenerate kernels were considered by many authors (see, for example [9-21]).

So, in our paper, using the method of degenerate kernel in combination it with the regularization method, we obtain an implicit functional-differential equation with nonlinear maxima. As you know, Fredholm functional integro-differential equation of first kind is ill-posed. So, we use initial boundary conditions to ensure the uniqueness of the solution. In order to use the method of a successive approximations, we transform the implicit functional-differential equation to the nonlinear Volterra type functional integro-differential equation, which is ill-posed, too. The one value solvability of this problem we have proved by the given initial boundary conditions.

On the segment $[0;T]$ the following nonlinear Fredholm integro-differential equation of first kind and first order is considered

$$\lambda\int_0^T K(t,s)F\left(s,u(s),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(s,u(s));h_2(s,u(s))\right]\right\},\int_0^s R(\theta)\dot u(\theta)\,d\theta\right)ds = f(t) \tag{1}$$

under the following conditions

$$\begin{cases} u(0)=\varphi_{01}=const,\\ \dot u(0)=\varphi_{02}=const,\\ u(t)=\varphi_1(t),\ t\in[-h_{01};0],\\ u(t)=\varphi_2(t),\ t\in[T;T+h_{02}], \end{cases} \tag{2}$$

where $0<T$ is given real number, λ is nonzero parameter of marching, $F(t,u,v,\vartheta)\in C([0;T]\times X\times X\times X)$, $h_i(t,u)\in C([0;T]\times X)$,

$-h_{01}<h_1(t,u)<h_2(t,u)<T+h_{02}$, $0<h_{0i}=const,\ i=1,2,$ $0<\int_0^t R(s)\,ds<\infty,$

$\varphi_1(t)\in C[-h_{01};0]$, $\varphi_2(t)\in C[T;T+h_{02}]$, $K(t,s)=\sum_{i=1}^k a_i(t)b_i(s)$,

$0\neq a_i(t),b_i(s)\in C[0;T]$, X is closed set on real number set. Here it is assumed that each of the systems of functions $a_i(t)$, $i=\overline{1,k}$, and $b_i(s)$, $i=\overline{1,k}$, linearly independent, $\varphi_1(0)=\varphi_{01}$, $\varphi_2(T)=u(T)$.

Method of degenerate kernel

Taking into account the degeneracy of the kernel, equation (1) is written in the following form

$$\lambda\int_0^T\sum_{i=1}^k a_i(t)b_i(s)F\left(s,u(s),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(s,u(s));h_2(s,u(s))\right]\right\},\int_0^s R(\theta)\dot u(\theta)\,d\theta\right)ds = f(t). \tag{3}$$

Using the notation

$$\vartheta(t)=F\left(t,u(t),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(t,u(t));h_2(t,u(t))\right]\right\},\int_0^t R(s)\dot{u}(s)\,d\,s\right) \tag{4}$$

and introducing new unknown function $\vartheta_\varepsilon(t)$, we obtain from (3) approximation Fredholm second kind integral equation with small parameter

$$\varepsilon\vartheta_\varepsilon(t)=f(t)-\lambda\int_0^T\sum_{i=1}^k a_i(t)\,b_i(s)\,\vartheta_\varepsilon(s)\,d\,s, \tag{5}$$

where

$$\lim_{\varepsilon\to 0}\vartheta_\varepsilon(t)=\vartheta(t), \tag{6}$$

$0<\varepsilon$ is small parameter.

Using the new notation

$$\alpha_i=\int_0^T b_i(s)\,\vartheta_\varepsilon(s)\,d\,s, \tag{7}$$

the integral equation (5) can be rewritten as follows

$$\vartheta_\varepsilon(t)=\frac{1}{\varepsilon}\left[f(t)-\lambda\sum_{i=1}^k a_i(t)\,\alpha_i\right]. \tag{8}$$

Substituting (8) into (7), we obtain the system of linear equations (SLE)

$$\alpha_i+\lambda\sum_{j=1}^k \alpha_j\,A_{ij}=B_i,\ \ i=\overline{1,k}, \tag{9}$$

where

$$A_{ij}=\frac{1}{\varepsilon}\int_0^T b_i(s)\,a_j(s)\,d\,s,\ \ B_i=\frac{1}{\varepsilon}\int_0^T b_i(s)\,f(s)\,d\,s. \tag{10}$$

Consider the following determinants:

$$\Delta(\lambda)=\begin{vmatrix} 1+\lambda A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & 1+\lambda A_{22} & \dots & A_{2k} \\ \dots & \dots & \dots & \dots \\ A_{k1}^m & A_{k2} & \dots & 1+\lambda A_{kk} \end{vmatrix}\neq 0, \tag{11}$$

$$\Delta_i(\lambda)=\begin{vmatrix} 1+\lambda A_{11} & \dots & A_{1(i-1)} & B_1 & A_{1(i+1)} & \dots & A_{1k} \\ A_{21} & \dots & A_{2(i-1)} & B_2 & A_{2(i+1)} & \dots & A_{1k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{k1} & \dots & A_{k(i-1)} & B_k & A_{k(i+1)} & \dots & 1+\lambda A_{kk} \end{vmatrix},\ \ i=\overline{1,k}.$$

SLE (9) is uniquely soluble for any finite right-hand sides, if the nondegeneracy condition (11) of the Fredholm determinant is fulfilled. The determinant $\Delta(\lambda)$ in (11) is a polynomial with respect to λ of degree not higher k. The equation $\Delta(\lambda)=0$ has at most k different real roots. We denote them by $\mu_l \, (l=\overline{1,p}\,, 1\le p\le k)$. Then $\lambda=\mu_l$ called irregular values of the second spectral parameter λ. Other values of the spectral parameter $\lambda\ne\mu_l$ are called regular. The solutions of SLE (9) for regular values of parameter λ are written as

$$\alpha_i=\frac{\Delta_i(\lambda)}{\Delta(\lambda)}, \quad i=\overline{1,k}. \tag{12}$$

Substituting (12) into (8), we obtain

$$\vartheta_\varepsilon(t)=\frac{1}{\varepsilon}\left[f(t)-\lambda\sum_{i=1}^{k}a_i(t)\frac{\Delta_i(\lambda)}{\Delta(\lambda)}\right]. \tag{13}$$

By virtue of formula (10), we suppose that

$$f(t)=\lambda\sum_{i=1}^{k}a_i(t)c_i\,, \quad c_i-\lambda\frac{\Delta_i(\lambda)}{\Delta(\lambda)}=\varepsilon C_i, \tag{14}$$

where $c_i, C_i=const, \quad i=\overline{1,k}.$

The parameter λ is marching parameter between free term function $f(t)$ and kernel of integral equation (1). So, we choose one of the regular values of the parameter λ such that the first of condition (14) is fulfilled. Then, taking into account limit passing formula (6), from (13) we obtain

$$\vartheta(t)=\lambda\sum_{i=1}^{k}C_i a_i(t). \tag{15}$$

Now the function $\vartheta(t)$ is known and defines by the formula (15). So, we solve the implicit functional equation (4). We rewrite this implicit equation as

$$G\left(t,u(t),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(t,u(t));h_2(t,u(t))\right]\right\},\int_0^t R(s)\dot{u}(s)\,ds\right)=0 \tag{16}$$

with given conditions (2), where $G=F-\vartheta$.

Transform into nonlinear Volterra type integral equation

In solving the implicit functional equation (16) we use the method of successive approximations in combination it with the method of compressing mapping. However, it is impossible to directly apply the method of successive approximations to the equation (16) with nonlinear deviation. Therefore, in this work we propose the following method. On the segment $[0;T]$ we take arbitrary positive defined and continuous function $K_0(t)$. We introduce the notation

$$\psi(t,s)=\int_s^t K_0(\theta)\,d\theta, \;\psi(t,0)=\psi(t), \;t\in[0;T].$$

It is obvious that $\psi(t,s)=\psi(t)-\psi(s)$. By the solution of equation (1) we mean a continuous function $u(t)$ on the segment $[0;T]$ that satisfies equation (1) with the given conditions (2) and the Lipschitz condition:

$$\max\left\{\left\| u(t)-u(s)\right\|;\left\|\dot{u}(t)-\dot{u}(s)\right\|\right\}\leq L_0\left|t-s\right|, \tag{17}$$
where $0<L_0=const,\ \left\|u(t)\right\|=\max\limits_{0\leq t\leq T}\left|u(t)\right|.$

We write the implicit equation (16) as

$$\dot{u}(t)+\int_0^t K_0(s)\dot{u}(s)\,ds=\dot{u}(t)+\int_0^t K_0(s)\dot{u}(s)\,ds+$$
$$+G\left(t,u(t),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(t,u(t));h_2(t,u(t))\right]\right\},\int_0^t R(s)\dot{u}(s)\,ds\right),\ t\in[0;T].$$

Hence, using resolvent of the kernel $\left[-K_0(s)\right]$, we obtain

$$\dot{u}(t)=\dot{u}(t)+$$
$$+\int_0^t K_0(s)\dot{u}(s)\,ds+G\left(t,u(t),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(t,u(t));h_2(t,u(t))\right]\right\},\int_0^t R(s)\dot{u}(s)\,ds\right)+$$
$$+\int_0^t K_0(s)\exp\left\{-\psi(t,s)\right\}\left\{-\dot{u}(s)+\int_0^s K_0(\theta)\dot{u}(\theta)\,d\theta-\right.$$
$$\left.-G\left(s,u(s),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(s,u(s));h_2(s,u(s))\right]\right\},\int_0^s R(\theta)\dot{u}(\theta)\,d\theta\right)\right\}ds. \tag{18}$$

Appling Dirichlet's formula to (18) (see [21]), we derive the following Volterra type nonlinear functional-integro-differential equation

$$\dot{u}(t)=\Im_1(t;\dot{u})\equiv\int_0^t H(t,s)\dot{u}(s)\,ds+\exp\{-\psi(t)\}\times$$
$$\times\left[\dot{u}(t)+G\left(t,u(t),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(t,u(t));h_2(t,u(t))\right]\right\},\int_0^t R(s)\dot{u}(s)\,ds\right)\right]+$$
$$+\int_0^t K_0(s)\exp\left\{-\psi(t,s)\right\}\left\{\dot{u}(t)-\dot{u}(s)+\right.$$
$$+G\left(t,u(t),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(t,u(t));h_2(t,u(t))\right]\right\},\int_0^t R(s)\dot{u}(s)\,ds\right)-$$
$$\left.-G\left(s,u(s),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(s,u(s));h_2(s,u(s))\right]\right\},\int_0^s R(\theta)\dot{u}(\theta)\,d\theta\right)\right\}ds, \tag{19}$$

where

$$H(t,s)=K_0(s)\exp\left\{-\psi(t,s)\right\}-\int_s^t K_0(\theta)\exp\left\{-\psi(t,\theta)\right\}d\theta. \tag{20}$$

By integrating functional-integro-differential equation (19) on the interval $(0;t)$ with initial condition $u(0)=\varphi_{01}$ we obtain the following functional-integro-differential equation

$$u(t)=\Im_2(t;u)\equiv\varphi_{01}+\int_0^t(t-s)H(t,s)\dot u(s)\,ds+$$
$$+\int_0^t\left[\dot u(s)+G\left(s,u(s),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(s,u(s));h_2(s,u(s))\right]\right\},\int_0^s R(\theta)\dot u(\theta)\,d\theta\right)\right]\times$$
$$\times\exp\{-\psi(s)\}\,ds+\int_0^t(t-s)K_0(s)\exp\{-\psi(t,s)\}\{\dot u(t)-\dot u(s)+$$
$$+G\left(t,u(t),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(t,u(t));h_2(t,u(t))\right]\right\},\int_0^t R(s)\dot u(s)\,ds\right)-$$
$$-G\left(s,u(s),\max\left\{u(\tau)\,|\,\tau\in\left[h_1(s,u(s));h_2(s,u(s))\right]\right\},\int_0^s R(\theta)\dot u(\theta)\,d\theta\right)\Bigg\}\,ds. \qquad (21)$$

Remark. The nonlinear functional-integro-differential equations (19) and (21) are ill-posed [Non2], so we will study it with given conditions (2). In addition, we consider the conditions (2) as $u(t-0)=u(t+0)$ at the points $t=0$ and $t=T$.

Let be fulfilled the conditions (11) and (14). Then, instead of the Fredholm functional-integro-differntial equation of first kind (1) we will study the Volterra type functional-integro-differential equations (19) and (21) with conditions (2).

Theorem. Let be fulfilled the conditions (17) and

1). $\left\|G\left(t,u(t),v(t),\vartheta(t)\right)\right\|\le M_0,\quad 0<M_0=const;$

2). $\left|G\left(t,u_1(t),v_1(t),\vartheta_1(t)\right)-G\left(t,u_2(t),v_2(t),\vartheta_2(t)\right)\right|\le$
$$\le L_1(t)\left(\left|u_1(t)-u_2(t)\right|+\left|v_1(t)-v_2(t)\right|+\left|\vartheta_1(t)-\vartheta_2(t)\right|\right);$$

3). $\left|h_i\left(t,u_1(t)\right)-h_i\left(t,u_2(t)\right)\right|\le L_{2i}(t)\left|u_1(t)-u_2(t)\right|,\;\; 0<L_{2i}(t)\in C[0;T],\; i=1,2;$

4). $\rho<1,$ where $\rho=\dfrac{1}{2}\max\limits_{0\le t\le T}\left[P_1(t)+P_2(t)+V_1(t)+V_2(t)\right],$
$$P_1(t)=L_1(t)\left[2+L_0\left(L_{21}(t)+L_{22}(t)\right)\right]Q(t,0),$$
$$P_2(t)=\int_0^t Q(t,s)\,ds+\left(1+L_1(t)\int_0^t R(s)\,ds\right)Q(t,0),$$
$$Q(t,s)=\exp\{-\psi(t)\}+2\int_0^t K_0(s)\cdot\exp\{-\psi(t,s)\}\,ds;$$
$$V_1(t)=L_1(t)\left[2+L_0\left(L_{21}(t)+L_{22}(t)\right)\right]\tilde Q(t),$$

$$V_2(t)=\int_0^t (t-s)Q(t,s)\,ds+\left(1+L_1(t)\int_0^t R(s)\,ds\right)\tilde{Q}(t),$$

$$\tilde{Q}(t)=\int_0^t \exp\{-\psi(s)\}\,ds+2\int_0^t (t-s)K_0(s)\exp\{-\psi(t,s)\}\,ds.$$

Then the nonlinear functional-integro-differential equation (21) with conditions (2) has a unique solution on the segment $[0;T]$.

Proof. We suppose that Picard iteration process for functional-integro-differential equations (19) and (21) is given by

$$\dot{u}_0(t)=\varphi_{02},\ t\in[0;T],\ \dot{u}_{n+1}(t)=\Im_1(t;\dot{u}_n),\ n\in N,\ \ t\in[0;T], \tag{22}$$

$$\left[\begin{array}{l} u_0(t)=\varphi_1(t),\ t\in[-h_1;0],\\ u_0(t)=\varphi_{01},\ t\in[0;T],\\ u_0(t)=\varphi_2(t),\ t\in[T;T+h_2], \end{array}\right. \quad \left[\begin{array}{l} u_{n+1}(t)=\varphi_1(t),\ t\in[-h_1;0],\\ u_{n+1}(t)=\Im_2(t;u_n),\ n\in N,\ \ t\in[0;T],\\ u_{n+1}(t)=\varphi_2(t),\ t\in[T;T+h_2], \end{array}\right. \tag{23}$$

First, we take estimate for the function $H(t,s)$, given by formula (20):

$$|\,H(t,s)\,|\le K_0(s)\cdot\exp\{-\psi(t,s)\}+2\int_s^t K_0(\theta)\exp\{-\psi(t,\theta)\}\,d\theta=Q(t,s). \tag{24}$$

It is obvious that the following estimate is true

$$\|\,\dot{u}_0(t)\,\|\le|\,\varphi_{02}\,|<\infty. \tag{25}$$

$$\|\,u_0(t)\,\|\le\max\left\{|\,\varphi_{01}\,|;\max_{-h_1\le t\le 0}|\,\varphi_1(t)\,|;\max_{T\le t\le T+h_2}|\,\varphi_2(t)\,|\right\}=\Delta_0<\infty. \tag{26}$$

By virtue of conditions of theorem and Picard processes (22) and (23), by using estimates (25) and (26), for the first approximations we obtain the estimates

$$|\,\dot{u}_1(t)\,|\le\int_0^t\|\,H(t,s)\,\|\cdot\|\,\dot{u}_0(s)\,\|ds+\exp\{-\psi(t)\}\times$$

$$\times\left[\|\,\dot{u}_0(t)\,\|+\left\|G\left(t,u_0(t),\max\left\{u_0(\tau)|\,\tau\in\left[h_1(t,u_0(t));h_2(t,u_0(t))\right]\right\},\int_0^t R(s)\dot{u}_0(s)\,d\,s\right)\right\|\right]+$$

$$+\int_0^t K_0(s)\exp\{-\psi(t,s)\}\times$$

$$\times\left[\|\,\dot{u}_0(t)-\dot{u}_0(s)\,\|+2\left\|G\left(t,u_0(t),\max\left\{u_0(\tau)|\,\tau\in\left[h_1(t,u_0(t));h_2(t,u_0(t))\right]\right\},\int_0^t R(s)\dot{u}_0(s)\,d\,s\right)\right\|\right]ds$$

$$\le|\,\varphi_{02}|\int_0^t Q(t,s)\,ds+\left(|\,\varphi_{02}\,|+M_0\right)\cdot\exp\{-\psi(t)\}+$$

$$+\int_0^t K_0(s)\cdot\exp\left\{-\psi(t,s)\right\}\left(L_0\left|t-s\right|+2M_0\right)ds\le$$

$$\le|\varphi_{02}|\int_0^t Q(t,s)\,ds+\Delta_{11}Q(t,0), \qquad (27)$$

where

$$\Delta_{11}=\max\left\{|\varphi_{02}|+M_0;\ L_0T+2M_0\right\};$$

$$|u_1(t)|\le\Delta_0+\int_0^t\|(t-s)H(t,s)\|\cdot\|\dot u_0(s)\|ds+\int_0^t\exp\{-\psi(s)\}\Big[\|\dot u_0(s)\|+$$

$$+\left\|G\left(s,u_0(s),\max\left\{u_0(\tau)|\,\tau\in\left[h_1(s,u_0(s));h_2(s,u_0(s))\right]\right\},\int_0^s R(\theta)\dot u_0(\theta)\,d\theta\right)\right\|\Big]ds+$$

$$+\int_0^t(t-s)K_0(s)\exp\left\{-\psi(t,s)\right\}\Big[\|\dot u_0(t)-\dot u_0(s)\|+$$

$$+2\left\|G\left(t,u_0(t),\max\left\{u_0(\tau)|\,\tau\in\left[h_1(t,u_0(t));h_2(t,u_0(t))\right]\right\},\int_0^t R(s)\dot u(s)\,ds\right)\right\|\Big]ds\le$$

$$\le\Delta_0+|\varphi_{02}|\int_0^t(t-s)Q(t,s)\,ds+\left(|\varphi_{02}|+M_0\right)\int_0^t\exp\{-\psi(s)\}\,ds+$$

$$+\int_0^t(t-s)K_0(s)\cdot\exp\left\{-\psi(t,s)\right\}\left(L_0\left|t-s\right|+2M_0\right)ds\le$$

$$\le\Delta_0+|\varphi_{02}|\int_0^t(t-s)Q(t,s)\,ds+\Delta_{11}\tilde Q(t), \qquad (28)$$

where

$$\tilde Q(t)=\int_0^t\exp\left\{-\psi(s)\right\}ds+2\int_0^t(t-s)K_0(s)\exp\left\{-\psi(t,s)\right\}ds.$$

By virtue of first and second conditions of theorem, analogously to estimates (27) and (28) for arbitrary difference of approximations we have

$$|\dot u_{n+1}(t)-\dot u_n(t)|\le\int_0^t\|H(t,s)\|\cdot\|\dot u_n(s)-\dot u_{n-1}(s)\|ds+$$

$$+\exp\{-\psi(t)\}\Big[\|\dot u_n(t)-\dot u_{n-1}(t)\|+L_1(t)\Big(\|u_n(t)-u_{n-1}(t)\|+$$

$$+\Big\|\max\left\{u_n(\tau)|\,\tau\in\left[h_1(t,u_n(t));h_2(t,u_n(t))\right]\right\}-$$

$$-\max\left\{u_{n-1}(\tau)|\,\tau\in\left[h_1(t,u_{n-1}(t));h_2(t,u_{n-1}(t))\right]\right\}\Big\|+\int_0^t R(s)\|\dot u_n(s)-\dot u_{n-1}(s)\|\,ds\Big)\Big]+$$

$$+2\int_0^t K_0(s)\exp\{-\psi(t,s)\}\Big[\|\dot u_n(s)-\dot u_{n-1}(s)\|+L_1(s)\Big(\|u_n(s)-u_{n-1}(s)\|+$$
$$+\Big\|\max\{u_n(\tau)|\,\tau\in[h_1(s,u_n(s));h_2(s,u_n(s))]\}-$$
$$-\max\{u_{n-1}(\tau)|\,\tau\in[h_1(s,u_{n-1}(s));h_2(s,u_{n-1}(s))]\}\Big\|+$$
$$+\int_0^s R(\theta)\|\dot u_n(\theta)-\dot u_{n-1}(\theta)\|d\theta\Big)\Big]ds. \qquad (29)$$

To continue estimate the norm in (29) we use condition (17) and third condition of the theorem. Then we have

$$\Big\|\max\{u_n(\tau)|\,\tau\in[h_1(s,u_n(s));h_2(s,u_n(s))]\}-$$
$$-\max\{u_{n-1}(\tau)|\,\tau\in[h_1(s,u_{n-1}(s));h_2(s,u_{n-1}(s))]\}\Big\|\le$$
$$\le\Big\|\max\{u_n(\tau)|\,\tau\in[h_1(s,u_n(s));h_2(s,u_n(s))]\}-$$
$$-\max\{u_{n-1}(\tau)|\,\tau\in[h_1(s,u_n(s));h_2(s,u_n(s))]\}\Big\|+$$
$$+\Big\|\max\{u_{n-1}(\tau)|\,\tau\in[h_1(s,u_n(s));h_2(s,u_n(s))]\}-$$
$$-\max\{u_{n-1}(\tau)|\,\tau\in[h_1(s,u_{n-1}(s));h_2(s,u_{n-1}(s))]\}\Big\|\le$$
$$\le\Big\|\max\{|u_n(\tau)-u_{n-1}(\tau)|:\tau\in[h_1(s,u_n(s));h_2(s,u_n(s))]\}\Big\|+$$
$$+L_0\sum_{i=1}^{2}\|h_i(s,u_n(s))-h_i(s,u_{n-1}(s))\|\le$$
$$\le\big[1+L_0\big(L_{21}(s)+L_{22}(s)\big)\big]\|u_n(s)-u_{n-1}(s)\|. \qquad (30)$$

Substituting (30) into (29), we obtain

$$|\dot u_{n+1}(t)-\dot u_n(t)|\le\int_0^t Q(t,s)\|\dot u_n(s)-\dot u_{n-1}(s)\|ds+\exp\{-\psi(t)\}\times$$
$$\times\Big[L_1(t)\big[2+L_0\big(L_{21}(t)+L_{22}(t)\big)\big]\|u_n(t)-u_{n-1}(t)\|+\Big(1+L_1(t)\int_0^t R(s)ds\Big)\|\dot u_n(t)-\dot u_{n-1}(t)\|\Big]+$$
$$+2\int_0^t K_0(s)\cdot\exp\{-\psi(t,s)\}\Big[L_1(s)\big[2+L_0\big(L_{21}(s)+L_{22}(s)\big)\big]\|u_n(s)-u_{n-1}(s)\|+$$
$$+\Big(1+L_1(s)\int_0^s R(\theta)d\theta\Big)\|\dot u_n(s)-\dot u_{n-1}(s)\|\Big]ds\le$$
$$\le P_1(t)\|u_n(t)-u_{n-1}(t)\|+P_2(t)\|\dot u_n(t)-\dot u_{n-1}(t)\|, \qquad (31)$$

where

$$P_1(t)=L_1(t)\big[2+L_0\big(L_{21}(t)+L_{22}(t)\big)\big]Q(t,0),$$

$$P_2(t)=\int_0^t Q(t,s)\,ds+\left(1+L_1(t)\int_0^t R(s)\,ds\right)Q(t,0),$$

$$Q(t,s)=\exp\{-\psi(t)\}+2\int_0^t K_0(s)\cdot\exp\{-\psi(t,s)\}\,ds;$$

$$\begin{aligned}
&|\,u_{n+1}(t)-u_n(t)\,|\le\int_0^t (t-s)Q(t,s)\|\,\dot u_n(s)-\dot u_{n-1}(s)\,\|ds+\\
&+\int_0^t \exp\{-\psi(s)\}\Big[L_1(s)\big[2+L_0\left(L_{21}(s)+L_{22}(s)\right)\big]\|\,u_n(s)-u_{n-1}(s)\,\|+\\
&+\left(1+L_1(s)\int_0^s R(\theta)\,d\theta\right)\|\,\dot u_n(s)-\dot u_{n-1}(s)\,\|\Big]ds+\\
&+2\int_0^t (t-s)K_0(s)\cdot\exp\{-\psi(t,s)\}\times\\
&\times\Big[L_1(s)\big[2+L_0\left(L_{21}(s)+L_{22}(s)\right)\big]\|\,u_n(s)-u_{n-1}(s)\,\|+\\
&+\left(1+L_1(s)\int_0^t R(s)\,ds\right)\|\,\dot u_n(s)-\dot u_{n-1}(s)\,\|\Big]ds\le\\
&\le V_1(t)\|\,u_n(t)-u_{n-1}(t)\,\|+V_2(t)\|\,\dot u_n(t)-\dot u_{n-1}(t)\,\|,
\end{aligned}\tag{32}$$

where

$$V_1(t)=L_1(t)\big[2+L_0\left(L_{21}(t)+L_{22}(t)\right)\big]\tilde Q(t),$$

$$V_2(t)=\int_0^t (t-s)Q(t,s)\,ds+\left(1+L_1(t)\int_0^t R(s)\,ds\right)\tilde Q(t),$$

$$\tilde Q(t)=\int_0^t \exp\{-\psi(s)\}\,ds+2\int_0^t (t-s)K_0(s)\exp\{-\psi(t,s)\}\,ds.$$

From the estimates (31) and (32) follows that

$$\|\,U_{n+1}(t)-U_n(t)\,\|\le\rho\cdot\|\,U_n(t)-U_{n-1}(t)\,\|,\tag{33}$$

where

$$\|\,U_{n+1}(t)-U_n(t)\,\|\le\max\left\{\|\,u_{n+1}(t)-u_n(t)\,\|;\ \|\,\dot u_{n+1}(t)-\dot u_n(t)\,\|\right\},$$

$$\rho=\frac{1}{2}\max_{0\le t\le T}\left[P_1(t)+P_2(t)+V_1(t)+V_2(t)\right].$$

In choosing the function $K_0(t)$ we take into account that

$$\psi(t,s)=\int_s^t K_0(\theta)\,d\theta\gg 1,\ t\in[0;T].$$

Hence, we obtain that $\exp\{-\psi(t)\}\ll 1$. So, the functions $H(t,s)$ and $Q(t,s)$ are small. Then the functions $L_1(t),\ L_{2i}(t),\ i=1,2$ we can choose such that $\rho<1$ and last condition of the theorem is fulfilled. We consider the solution of the integral equations (19) and (21) in the space of continuous functions $C[0;T]$, satisfying condition (17). Since $\|u_{n+1}(t)-u_n(t)\|\leq\|U_{n+1}(t)-U_n(t)\|$, it follows from the estimate (33) that the integral operator on the right-hand side of (21) with conditions (2) is compressing mapping. So, from the estimates (25)--(28) and (33) implies that the integral equation (21) with conditions (2) has a unique solution on the segment $[0;T]$. The theorem is proved.

Conclusion

In this paper, we studied the problems of one value solvability and construction of solutions of a nonlinear Fredholm first kind functional integro-differential equation (1) of first order with degenerate kernel and nonlinear maxima. This Fredholm functional-integro-differential equation is ill-posed. So, we use boundary conditions (2) to ensure the uniqueness of the solution. First, using the method of degenerate kernel, we obtained the implicit functional-differential equation (16). In order to use the method of a successive approximations we reduce the implicit functional-differential equation (16) with nonlinear maxima to the nonlinear Volterra type first order functional integro-differential equation. So, the feature of this paper is such that first kind nonlinear Fredholm functional integro-differential equation (1) was replaced by the Volterra type functional integro-differential equation (21).

The nonlinear functional integro-differential equation (21) we conditionally called as a second kind Volterra type nonlinear functional integro-differential equation of first order. Because this Volterra type integro-differential equation (21) is ill-posed, too. So, we studied it by the given conditions (2). In addition, in the conditions (2) we suppose the continuous gluing conditions that $u(t-0)=u(t+0)$ at the points $t=0$ and $t=T$.

Let be fulfilled the conditions (11) and (14). Then, instead of the first kind Fredholm functional integro-differential equation of first order (1) we will study the second kind Volterra type functional integro-differential equation of first order (21) with conditions (2). The theorem of one value solvability of the problem (1), (2) was proved.

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